Partial Solution Set, Leon §6.7

6.7.1 For each of the following, compute the determinants of all of the leading principal submatrices and use them to determine whether the matrix is positive definite. (In other words, compute the leading principal minors.)

1. $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Leading principal minors are 2 and 3; positive definite.
2. $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$. Leading principal minors are 3, -7; not positive definite.
3. $A = \begin{bmatrix} 6 & 4 & -2 \\ 4 & 5 & 3 \\ -2 & 3 & 6 \end{bmatrix}$. Leading principal minors are 6, 14, -38; not positive definite.
4. $A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & -2 \\ 1 & -2 & 5 \end{bmatrix}$. Leading principal minors are 4, 8, and 13; positive definite.

6.7.3 We are given a 4×4 real symmetric matrix A, and asked to find its LU decomposition. As it turns out, this can be done without row interchanges, and all pivots are positive. It follows that A is positive definite.

6.7.5 For each of the following, find the Cholesky decomposition $A = BB^T$, where B is a lower triangular matrix. (Problem 6.7.4 is to find factorizations of the form LDL^T, which is an intermediate step here.)

1. $A = \begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 9 & 3 & -6 \\ 3 & 4 & 1 \\ -6 & 1 & 9 \end{bmatrix}$.

Solution:

1.

$$\begin{aligned}
A &= \begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 0 & 9 \end{bmatrix} \ (LU) \\
&= \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \ (LDL^T) \\
&= \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \ (BB^T).
\end{aligned}$$

(d)

$$\begin{aligned}
A &= \begin{bmatrix} 9 & 3 & -6 \\ 3 & 4 & 1 \\ -6 & 1 & 9 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 3 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix} \ (LU) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1/3 & -2/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \ (LDL^T) \\
&= \begin{bmatrix} 3 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ -2 & \sqrt{3} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & \sqrt{2} \end{bmatrix} \ (BB^T)
\end{aligned}$$

6.7.8 Let A be a symmetric positive definite matrix, and let Q be an orthogonal diagonalizing matrix for A. Use the factorization $A = QDQ^T$ to find a nonsingular matrix B such that $B^T B = A$.

Solution: Let $C = D^{1/2}$, and let $B = CQ^T$. Then

$$B^T B = \left(CQ^T\right)^T CQ^T = QC^T CQ^T = QDQ^T = A.$$

6.7.9 Let B be an $m \times n$ matrix of rank n. Show that $B^T B$ is positive definite.

Proof: We show that all eigenvalues of $B^T B$ are positive. First suppose that λ is an eigenvalue of $B^T B$, with associated eigenvector $\mathbf{x}$. Then

$$\begin{aligned}
\lambda \|\mathbf{x}\|_2^2 &= \lambda \mathbf{x}^T \mathbf{x} \\
&= \mathbf{x}^T \lambda \mathbf{x} \\
&= \mathbf{x}^T B^T B \mathbf{x} \\
&= (B\mathbf{x})^T B\mathbf{x} \\
&= \|B\mathbf{x}\|_2^2 \\
&\geq 0,
\end{aligned}$$

and it follows that $\lambda \geq 0$. We must now show that $\lambda = 0$ is impossible. Could $\lambda = 0$? This would signal that $B^T B$ has a nontrivial nullspace. But we know, from previous work, that if B has full column rank then $B^T B$ is nonsingular and so has the trivial nullspace. □

6.7.13 Let A be an $n \times n$ negative definite matrix.

1. What will be the sign of $\det(A)$ if n is even? If n is odd?
2. Show that the leading principal submatrices of A are negative definite.
3. Show that the determinants of the leading principal submatrices (the leading principal minors) alternate in sign.

Solution:

1. Since the determinant is the product of the eigenvalues, and since all of the eigenvalues of A are negative, it follows that $\det(A)$ will be positive if n is even, negative if n is odd.
2. Since A is negative definite, then $-A$ is positive definite. We know that the leading principal submatrices of $-A$ must be positive definite. But if $-A_r$ is the rth leading principal submatrix of $-A$, then A_r is the rth leading principal submatrix of A; since $-A_r$ is positive definite, then A_r is negative definite. The result follows.
3. By (b), the leading principal submatrices of A are negative definite. By (a), they must alternate in sign.